Abstract Algebra I
Exercise Sheet 1: Groups

Binary Operations

Question 1. For the following, give the conditions, if they exist, that must be satisfied for $*$ to be a binary operation:

- $* : S \times S \times S \times S \to S : (s_1, s_2, s_3, s_4) \mapsto s_1 + s_2 + s_3 + s_4,\ s_i \in S$;
- $* : S \times S \times S \times S \to S \times S : (s_1, s_2, s_3, s_4) \mapsto (s_1 + s_2, s_3 + s_4),\ s_i \in S$;
- $* : S \times S \to S : (s_1, s_2) \mapsto s_3,\ s_i \in S$.

Question 2. Is division a binary operation on $\mathbb{Z}$? What about on $\mathbb{Q}$?

Groups

For each of the following determine whether G is a group.

Question 3. $G = (\mathbb{Z}, +)$.

Question 4. $G = (\mathbb{Z}, \cdot)$.

Question 5. $G = (\mathbb{Z}_n, +) = \big(\{0, 1, \ldots, n-1\}, +\big)$.

Question 6. $G = (\mathbb{Z}_n, +) = \big(\{1, \ldots, n\}, \cdot\big)$.

Question 7. $G = (\mathbb{R}, +)$.

Question 8. $G = (\mathbb{R}, \cdot)$.

Question 9. $G = (\mathbb{R} \setminus \{0\}, \cdot)$.

Question 10. $G = \big(M_{n,\mathbb{R}}, +\big)$, where $M_{n,\mathbb{R}}$ is the set of $n \times n$ matrices with real entries.

Question 11. $G = \big(M_{n,\mathbb{R}}, \cdot\big)$.

Question 12. $G = \big(M_{n,\mathbb{Z}}, +\big)$, where $M_{n,\mathbb{Z}}$ is the set of $n \times n$ matrices with integer entries.

Question 13. $G = \big(M_{n,\mathbb{Z}}, \cdot\big)$.

Question 14. $G = \big(\mathrm{SL}_n(\mathbb{Z}), +\big)$, where $\mathrm{SL}_n(\mathbb{Z})$ is the set of $n \times n$ matrices with integer entries and determinant 1.

Question 15. $G = \big(\mathrm{SL}_2(\mathbb{Z}), \cdot\big)$.

Question 16. Let S be a set.

- $G = \Big(\big\{f : S \to S \,|\, f \text{ injective}\big\}, \circ\Big)$.
- $G = \Big(\big\{f : S \to S \,|\, f \text{ surjective}\big\}, \circ\Big)$.
- $G = \Big(\big\{f : S \to S \,|\, f \text{ bijective}\big\}, \circ\Big)$.